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Harmonic and pulse excitations of multiply connected cylindrical bodies☆

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Abstract

The three-dimensional problem of the theory of elasticity of the harmonic oscillations of cylindrical bodies (a layer with several tunnel cavities on a cylinder of finite length) is considered for uniform mixed boundary conditions on its bases. Using the Φ-solutions constructed, the boundary-value problems are reduced to a system of well-known one-dimensional singular integral equations. The solution of the problem of the pulse excitation of a layer on the surface of a cavity is "assembled" from a packet of corresponding harmonic oscillations using an integral Fourier transformation with respect to time. The results of calculations of the dynamic stress concentration in a layer (a plate) weakened by one and two openings of different configuration are given, as well as the amplitude-frequency characteristics for a cylinder of finite length with a transverse cross section in the form of a square with rounded corners, and data of calculations for a trapeziform pulse, acting on the surface of a circular cavity, are presented.

Methods of homogeneous solutions[1] or superposition[2,3] and some others[4,5] have been effectively used to investigate the harmonic oscillations of a finite circular cylinder or a layer with a cavity. If there is a plane tunnel crack or inclusion-type nonuniformity in the layer, the solution can, in principle, be obtained using the method of homogeneous solutions in combination with an integral Fourier transformation.[6] A corresponding theory has been developed for multiple plane cracks, parallel to the bases of the layer.[7,8] In the case of configurations differing from circular, it is more convenient to use the method of integral equations. In this case the problem arises of the correspondence between the boundary conditions of the theory of elasticity and the boundary conditions for the set of metaharmonic functions, which occur in the corresponding homogeneous solutions (see Ref. 9).

A one-to-one correspondence between the densities in the integral representations of metaharmonic functions and the physical quantities – the jumps in the kinematic quantities on the surface of a cylindrical nonuniformity, was obtained in Ref. 10 by another method. However, this approach leads to the need to regularize divergent integrals and, as a consequence, leads to integro-differential equations of fairly complex structure.[10,11]

Below we develop a new approach to investigating the harmonic oscillations of multiply connected cylindrical bodies of fairly arbitrary configurations.

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1. Formulation of the problem

In a rectilinear Cartesian system of coordinates $Ox_1x_2x_3$ we will consider a uniform elastic isotropic layer $-\infty < x_1$, $x_2 < \infty$, $|x_3| \le h$, weakened by tunnel cavities with a common boundary of the transverse cross-section $\Gamma = \cup\Gamma_\nu (\cap\Gamma_\nu = \emptyset$, $\nu = 1, 2, \ldots, N)$. We will assume that Γ_ν are simple closed contours without points of self-intersection, with Hölder-continuous curvatures. We will specify the stress vector $(X_{1n}, X_{2n}, X_{3n})(x, t)$, $x = (x_1, x_2, x_3) \in S$ on the surfaces of the cavities $S = \cup S_\nu$. We will take the following homogeneous boundary conditions of the mixed type on the bases of the layer

$$u_1 = u_2 = \sigma_{33} = 0, \quad x_3 = \pm h, \quad t > 0 \tag{1.1}$$

The problem consists of the determining the wave fields of the displacement vector $u = (u_1, u_2, u_3)$ and the stress tensor with components $\sigma_{ij} (i, j = 1, 2, 3)$ for harmonic or pulse excitation of the layer.

To determine the wave field of the displacements we will use the Lamé system of equations

$$\begin{gathered}\Delta u_j + \sigma\partial_j\vartheta + \frac{X_j}{\mu} = \frac{\rho}{\mu}\frac{\partial^2 u_j}{\partial t^2}, \quad j = 1, 2, 3 \\ \partial_k = \frac{\partial}{\partial x_k}, \quad \Delta = \partial_k\partial_k, \quad \vartheta = \partial_k u_k, \quad \sigma = \frac{1}{1-2\nu}\end{gathered} \tag{1.2}$$

where Δ is the Laplace operator in R^3, ϑ is the volume expansion, X_j is the strength of the bulk forces, μ and ν are the shear modulus and Poisson's ratio, and ρ is the density of the material. Here and henceforth summation is carried out over repeated subscripts i, k from 1 to 3.

We will introduce the following notation

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}, \quad \gamma_l = \frac{\omega}{c_l}, \quad \mu_{lm}^2 = \gamma_l^2 - \lambda_m^2, \quad \Delta_{lm} = \partial_1^2 + \partial_2^2 + \mu_{lm}^2; \quad l = 1, 2$$

where c_1 and c_2 are the propagation velocities of a longitudinal wave and a transverse (shear) wave in the elastic medium, and γ_1 and γ_2 are the corresponding wave numbers.

We will first consider harmonic excitation of the layer; we will put

$$u_j = e^{-i\omega t}U_j, \quad \vartheta = e^{-i\omega t}\theta = e^{-i\omega t}\partial_k U_k, \quad X_j = e^{-i\omega t}Y_j, \quad j = 1, 2, 3 \tag{1.3}$$

where $U_j = U_j(x)$, $\theta = \theta(x)$, $Y_j = Y_j(x)$ $(x = (x_1, x_2, x_3))$ are the amplitudes of the corresponding quantities.

Eliminating the time t in Eqs. (1.2) using the representations (1.3), we arrive at a system of differential equations in the amplitudes

$$\Delta U_j + \sigma\partial_j\partial_k U_k + \gamma_2^2 U_j = -\frac{Y_j}{\mu}, \quad j = 1, 2, 3 \tag{1.4}$$

It is necessary to add to system (1.4) the following boundary conditions on the surfaces of the cavities

$$S_{ij}n_i = Y_{jn}, \quad j = 1, 2, 3 \tag{1.5}$$

where S_{ij} and Y_{jn} are the amplitude values of the quantities σ_{ij} and X_{jn} respectively.

Bearing in mind the symmetrical state relative to the middle plane of the layer, we can represent the amplitudes of the displacements and the strengths of the bulk forces in the form of Fourier series

$$\{U_1, U_2, \theta, Y_1, Y_2\} = \sum_{m=1}^{\infty}\{U_{1m}, U_{2m}, \theta_m Y_{1m}, Y_{2m}\}\cos\lambda_m x_3$$

$$\{U_3, Y_3\} = \sum_{m=1}^{\infty}\{U_{3m}, Y_{3m}\}\sin\lambda_m x_3 \tag{1.6}$$

$$U_{jm} = U_{jm}(x_1, x_2), \quad Y_{jm} = Y_{jm}(x_1, x_2); \quad j = 1, 2, 3$$

$$\theta_m = \partial_1 U_{1m} + \partial_2 U_{2m} + \lambda_m U_{3m}, \quad \lambda_m = \pi(2m-1)/(2h)$$

In this case boundary conditions (1.1) on the bases of the layer will be satisfied.

Eliminating the thickness coordiate x_3 in Eqs. (1.4) using representations (1.6), we arrive at a system of equations in the Fourier coefficients U_{jm}

$$\Delta_{2m}U_{lm} + \sigma\partial_l\theta_m = -\frac{Y_{lm}}{\mu}, \quad l = 1, 2; \quad \Delta_{2m}U_{3m} - \sigma\lambda_m\theta_m = -\frac{Y_{3m}}{\mu}; \quad m = 1, 2, \ldots \tag{1.7}$$

To eliminate the thickness coordinate from Eqs. (1.5), we will use the following representations for the amplitudes of the components of the stress tensor and the surface-load vector, consistent with representations (1.6)

$$\{S_{uv}, Y_{un}\} = \sum_{m=1}^{\infty}\{S_{uv}^{(m)}, Y_{un}^{(m)}\}\cos\lambda_m x_3, \quad u, v = 1, 2$$

$$\{S_{j3}, Y_{3n}\} = \sum_{m=1}^{\infty}\{S_{j3}^{(m)}, Y_{3n}^{(m)}\}\sin\lambda_m x_3, \quad j = 1, 2, 3 \tag{1.8}$$

Then the boundary conditions (1.5) decompose into a set of equalities of the form

$$S_{ij}^{(m)}n_i = Y_{jn}^{(m)}, \quad j = 1, 2, 3; \quad m = 1, 2, \ldots \tag{1.9}$$

2. The Φ-solutions for a layer

Suppose now that forces with strengths $\{P_1, P_2, P_3\}(x_3)$ per unit length are distributed along the cord $x_1 = 0, x_2 = 0$, $|x_3| \le h$. Then, the Fourier coefficients of the strengths of the bulk forces, which occur on the right-hand sides of Eqs. (1.7), take the form

$$Y_{jm} = P_{jm}\delta(x), \quad x = (x_1, x_2), \quad j = 1, 2, 3 \tag{2.1}$$

where $\delta(x)$ is the two-dimensional delta function.

We mean by the Φ-solutions for the layer, corresponding to mixed boundary conditions (1.1), the components of the matrix of the fundamental solutions of system (1.7) with right-hand sides defined by Eqs. (2.1).

From Eqs. (1.7) we derive by the usual method

$$\Delta_{1m}\theta_m = -\frac{1}{\mu(1+\sigma)}(P_{1m}\partial_1 + P_{2m}\partial_2 + P_{3m}\lambda_m)\delta(x) \tag{2.2}$$

We will consider in more detail the case when $P_1 \ne 0$ and $P_2 = P_3 = 0$.

From Eq. (2.2) for the case considered we obtain the inhomogeneous Helmholtz equation

$$\Delta_{1m}\theta_m^{(1)} = -\frac{P_{1m}}{\mu(1+\sigma)}\partial_1\delta(x) \tag{2.3}$$

Suppose E is the fundamental solution of the Helmholtz operator.[12] Taking into account the fact that $\delta(x)$ is finite and the convolution $E \times \partial_1 f = f \times \partial_1 E$ exists, we obtain from Eq. (2.3)

$$\theta_m^{(1)} = \frac{iP_{1m}}{4\mu(1+\sigma)}\partial_1 H_{01m} \tag{2.4}$$

where

$$H_{plm} = H_p^{(1)}(\mu_{lm}r), \quad r = \sqrt{x_1^2 + x_2^2}, \quad p = 0, 1, 2, 3; \quad l = 1, 2$$

and $H_p^{(1)}(x)$ is the Hankel function of the first kind of order p.

Formula (2.4) enables us to separate the equations in system (1.7) and to represent it in the form

$$\begin{aligned}\Delta_{2m}U_{1m}^{(1)} &= -\frac{i\sigma P_{1m}}{4\mu(1+\sigma)}\partial_1^2 H_{01m} - \frac{P_{1m}}{\mu}\delta(x)\\ \Delta_{2m}U_{2m}^{(1)} &= -\frac{i\sigma P_{1m}}{4\mu(1+\sigma)}\partial_1\partial_2 H_{01m}, \quad \Delta_{2m}U_{3m}^{(1)} = \frac{i\sigma P_{1m}}{4\mu(1+\sigma)}\lambda_m\partial_1 H_{01m}\end{aligned} \tag{2.5}$$

Integration of system (2.5) in the space of generalized functions $\mathcal{D}'(R^2)$ gives

$$U_{1m}^{(1)} = \frac{iP_{1m}}{4\mu}\left(-\frac{1}{\gamma_2^2}\partial_1^2 H + H_{02m}\right), \quad U_{2m}^{(1)} = -\frac{iP_{1m}}{4\mu\gamma_2^2}\partial_1\partial_2 H, \quad U_{3m}^{(1)} = \frac{iP_{1m}}{4\mu\gamma_2^2}\partial_1\lambda_m H$$

$$H = H_{01m} - H_{02m}$$

In a similar way we can consider the case when $P_2 \neq 0$, $P_1 = P_3 = 0$ and $P_3 \neq 0$, $P_1 = P_2 = 0$. We will write the final results for the displacement vector

$$U_{nm}^{(j)} = \frac{iP_{jm}}{4\mu}g_{nm}^{(j)} \tag{2.6}$$

The quantities $g_{nm}^{(j)}$ are the components of the matrix of the Φ-solutions for each fixed value of m

$$g_m = \left\|g_{nm}^{(j)}\right\|, \quad n, j = 1, 2, 3; \quad m = 1, 2, \ldots$$

$$\begin{aligned}g_{\upsilon m}^{(\upsilon)} &= \frac{1}{2\gamma_2^2}\sum_{l=1}^{2}(-1)^{l+1}\mu_{lm}^2((-1)^{\upsilon}H_{2lm}\cos 2\alpha + H_{0lm}) + H_{02m}, \quad \upsilon = 1, 2\\ g_{2m}^{(1)} &= g_{1m}^{(2)} = -\frac{1}{2\gamma_2^2}\sum_{l=1}^{2}(-1)^{l+1}\mu_{lm}^2 H_{2lm}\sin 2\alpha\\ \frac{g_{3m}^{(1)}}{\cos\alpha} &= -\frac{g_{1m}^{(3)}}{\cos\alpha} = \frac{g_{3m}^{(2)}}{\sin\alpha} = -\frac{g_{2m}^{(3)}}{\sin\alpha} = -\frac{\lambda_m}{\gamma_2^2}\sum_{l=1}^{2}(-1)^{l+1}\mu_{lm}H_{1lm}\\ g_{3m}^{(3)} &= \frac{\lambda_m^2}{\gamma_2^2}H + H_{02m}\end{aligned} \tag{2.7}$$

The actual values of the displacements for the general case can be found from the formulae

$$u_l = \mathrm{Re}\left(e^{-i\omega t}\sum_{j=1}^{3}\sum_{m=1}^{\infty}U_{lm}^{(j)}\cos\lambda_m x_3\right), \quad l = 1, 2; \quad u_3 = \mathrm{Re}\left(e^{-i\omega t}\sum_{j=1}^{3}\sum_{m=1}^{\infty}U_{3m}^{(j)}\sin\lambda_m x_3\right) \tag{2.8}$$

Expressions (2.6) and (2.8) provide a representation of the waveguide properties of the layer. It can be seen that for any excitation frequency, there is always a number m for which the characteristic number μ_{1m} or both characteristic numbers become pure imaginary, which leads to non-uniform waves, decaying exponentially with r. When $\pi(2m-1)<2\gamma_1 h$ the first m terms in series (1.6) are the superposition of waves propagating from the source. Terms of the series which satisfy the inequality $\pi(2m-1)>2\gamma_2 h$, decay exponentially as r increases, and also as the number m increases. Hence it follows that the thicker the waveguide the wider the frequency spectrum transmitted.

It can be seen from this that the residues of series (1.6) approach zero, since the common term of any of the series decreases exponentially as the number m increases. It can be shown that the series converge absolutely when $r\neq 0$.

3. Integral representations of the solutions of boundary-value problem (1.7), (1.9)

Suppose $f\in C^2(\bar{G})$, where $G=R^2\Gamma$ is a physical field with a line of discontinuity Γ. We will write formulae for the generalized derivative[12]

$$\partial_j f = \{\partial_j f\} + n_j[f]\delta_\Gamma \quad (\partial_1^2+\partial_2^2)f = \{(\partial_1^2+\partial_2^2)f\} + \left[\frac{\partial f}{\partial n}\right]\delta_\Gamma + \frac{\partial}{\partial n}([f]\delta_\Gamma)$$

where $\{\cdot\}$ is the corresponding classical derivative, $[\cdot]$ is the jump in the function indicated on the contour Γ, n_j is the projection of the unit vector of the normal to the contour Γ onto the x_j axis, and $[\cdot]\delta_\Gamma$ and $\frac{\partial}{\partial n}([\cdot]\delta_\Gamma)$ are simple and double layers respectively.

Introducing these relations into system (1.7), taking into account expressions (2.1), we can represent it in the form

$$\begin{aligned} &\Delta_{2m}U_{jm} + \sigma\partial_j\theta_m = f_{jm}, \quad j = 1,2,3; \quad m = 1,2,\dots \\ &f_{jm} = -\left[\frac{\partial U_{jm}}{\partial n}\right]\delta_\Gamma - \frac{\partial}{\partial n}([U_{jm}]\delta_\Gamma) - \sigma[\theta_m]n_j\delta_\Gamma, \quad n_3 = 0 \end{aligned} \tag{3.1}$$

Using the matrix of the Φ-solutions (2.7), the solution of system (3.1) can be represented in the form of a convolution

$$U_m(x) = \{U_{1m}, U_{2m}, U_{3m}\} = g_m * f_m; \quad x = (x_1, x_2), \quad f_m = \{f_{1m}, f_{2m}, f_{3m}\}$$

In expanded form, we hence obtain integral representations of the wave field of the displacements (everywhere henceforth, unless otherwise stated, the integration is carried out over the contour Γ)

$$\begin{aligned} U_{jm}(x) &= \int [U_{km}](y)\frac{\partial}{\partial n_y}g_{jm}^{(k)}(x-y)dS_y \\ &-\int\left(\left[\frac{\partial U_{km}}{\partial n}\right] + \sigma[\theta_m]n_k\right)(y)g_{jm}^{(k)}(x-y)dS_y, \quad j = 1,2,3; \quad m = 1,2,\dots \end{aligned} \tag{3.2}$$

where dS_y is an element of the arc of the contour Γ and the summation is carried out over $k=1, 2, 3$.

In the case when the contour Γ is a set of closed arcs (mathematical sections) $\Gamma_\nu(\nu=1,2,\dots,N)$ and the stress vector can be extended continuously over the whole of Γ_ν, it is sufficient to retain the first term on the right-hand side of Eq. (3.2), to find the solution in the form of generalized potentials of the double layer. To solve the problem considered here, we will retain only the second term and we will seek the solution in the form of generalized potentials of the simple layer, which, in expanded form, can be written as

$$\begin{aligned} &U_{1m}(z) = \frac{1}{\gamma_2^2}\int\left[D_m\frac{\partial}{\partial\xi_1}H + p_m\gamma_2^2 H_{02m}\right]dS, \quad U_{2m}(z) = \frac{1}{\gamma_2^2}\int\left[D_m\frac{\partial}{\partial\xi_2}H + q_m\gamma_2^2 H_{02m}\right]dS \\ &U_{3m}(z) = \frac{1}{\gamma_2^2}\int[\lambda_m D_m H + r_m\gamma_2^2 H_{02m}]dS, \quad \theta_m(z) = \int D_m H_{01m}\frac{dS}{1+\sigma} \\ &D_m = -p_m\frac{\partial}{\partial\xi_1} - q_m\frac{\partial}{\partial\xi_2} + r_m\lambda_m \end{aligned} \tag{3.3}$$

The functions

$$p_m = \{p_m^\nu(\zeta), \zeta\in\Gamma_\nu\}, \quad q_m = \{q_m^\nu(\zeta), \zeta\in\Gamma_\nu\}, \quad r_m = \{r_m^\nu(\zeta), \zeta\in\Gamma_\nu\}$$

remain to be determined, $\zeta = \xi_1 + i\xi_2 \in \Gamma = \cup\Gamma_\nu$, dS is an element of the arc of the contour Γ, $\zeta - z = re^{i\alpha}$, and the function $H = H(r)$ is defined in Section 2.

4. The resolving system of integral equations

It is convenient to represent boundary conditions (1.9) on the contour Γ in complex form

$$\begin{aligned} &S_1^{(m)} - e^{\pm 2i\psi}S_2^{\pm(m)} = 2e^{i\psi}(Y_1^{(m)} \mp iY_2^{(m)}) = 2(N^{(m)} \mp iT^{(m)}) \\ &e^{i\psi}S_3^{-(m)} + e^{-i\psi}S_3^{+(m)} = 2Y_3^{(m)}; \quad m = 1, 2, \dots \\ &S_1^{(m)} = S_{11}^{(m)} + S_{22}^{(m)}, \quad S_2^{\pm(m)} = S_{22}^{(m)} - S_{11}^{(m)} \pm 2iS_{12}^{(m)}, \quad S_3^{\pm(m)} = S_{13}^{(m)} \pm iS_{23}^{(m)} \end{aligned} \tag{4.1}$$

where ψ is the angle between the normal to the contour Γ and the Ox_1 axis, and $N^{(m)}$ and $T^{(m)}$ are the Fourier coefficients of the amplitudes of the normal and shear stresses on Γ.

Using Hooke's law in amplitudes, we obtain representations of the combinations introduced in (4.1) in terms of the components of the displacement vector

$$\begin{aligned} &S_1^{(m)} = 2\mu(\sigma\theta_m - \lambda_m U_{3m}), \quad S_2^{\pm(m)} = -2\mu(\partial_1 \mp i\partial_2)\frac{\partial}{\partial z}V_m^{\mp} \\ &S_3^{\pm(m)} = \mu\{(\partial_1 \mp i\partial_2)U_{3m} - \lambda_m V_m^{\mp}\}; \quad V_m^{\pm} = U_{1m} \pm iU_{2m} \end{aligned} \tag{4.2}$$

We will introduce the functions y_{jm} by the equalities

$$p_m = y_{1m}e^{i\psi} + y_{2m}e^{-i\psi}, \quad q_m = i(y_2e^{-i\psi} - y_{1m}e^{i\psi}), \quad r_m = y_{3m} \tag{4.3}$$

Substituting the limit values of the combinations (4.2) into the boundary Eqs. (4.1) using representations (3.3) and bearing in mind formulae (4.3), we obtain a system of singular integral equations of the boundary-value problems (1.7), (1.9)

$$\mp iy_{jm}(\zeta_0) + \frac{1}{4}\int y_{km}(\zeta)K_{jk}dS = \frac{1}{4\mu}\chi_j^{(m)}, \quad j = 1, 2, 3 \tag{4.4}$$

Here

$$\chi_1^{(m)} = (Y_1^{(m)} + iY_2^{(m)})e^{-i\psi_0}, \quad \chi_2^{(m)} = (Y_1^{(m)} - iY_2^{(m)})e^{i\psi_0}, \quad \chi_3^{(m)} = 2Y_3^{(m)}$$

$$K_{11} = \left[\frac{1}{2(1-\nu)}\mu_{1m}H_{11m}^0 - \lambda_m^2 g_{1m} + \{g_{4m} + 2\mu_{2m}H_{12m}^0\}e^{2i(\alpha_0 - \psi_0)}\right]e^{i(\psi - \alpha_0)}$$

$$K_{12} = \left[\frac{1}{2(1-\nu)}\mu_{1m}H_{11m}^0 - \lambda_m^2 g_{1m} - g_{3m}e^{2i(\alpha_0 - \psi_0)}\right]e^{i(\alpha_0 - \psi)}$$

$$K_{13} = \lambda_m\left[\frac{1}{2(1-\nu)}H_{01m}^0 - H_{02m}^0 - \frac{\lambda_m^2}{\gamma_2^2}H^0 - g_{2m}e^{2i(\alpha_0 - \psi_0)}\right]$$

$$K_{31} = 2\lambda_m[g_{2m}e^{2i(\psi_0 - \alpha_0)} - g_{0m} - H_{02m}^0]e^{i(\psi - \psi_0)}$$

$$K_{33} = 2[2\lambda_m^2 g_{1m} + \mu_{2m}H_{12m}^0]\cos(\alpha_0 - \psi_0)$$

$$\zeta_0 \in \Gamma = \cup\Gamma_\nu, \quad \psi_0 = \psi(\zeta_0), \quad \zeta - \zeta_0 = r_0e^{i\alpha_0}$$

$$g_{0m} = \frac{1}{\gamma_2^2}\sum_{l=1}^{2}(-1)^{l+1}\mu_{lm}^2 H_{0lm}^0, \quad g_{jm} = \frac{1}{\gamma_2^2}\sum_{l=1}^{2}(-1)^{l+1}\mu_{lm}^j H_{jlm}^0, \quad j = 1, 2, 3$$

$$g_{4m} = \frac{1}{\gamma_2^2}\sum_{l=1}^{2}(-1)^{l+1}\mu_{lm}^3 H_{1lm}^0, \quad H_{plm}^0 = H_p^{(1)}(\mu_{lm}r_0), \quad H^0 = H(r_0)$$

$$p = 0, 1, 2, 3; \quad l = 1, 2$$

The kernels K_{22}, K_{21}, K_{23}, K_{32} are obtained from K_{11}, K_{21}, K_{13}, K_{31} by replacing α_0, ψ_0, ψ by $-\alpha_0$, $-\psi_0$, $-\psi$ respectively.

The resultant index of system (4.4) is equal to zero; consequently, it is uniquely solvable for any frequency ω not belonging to the spectrum.

Remark. System (4.4) can be used both to investigate wave fields in a layer (plate) with cavities, retaining the lower sign in terms outside the integral, and when considering the oscillations of cylinders of finite length (the upper sign).

An expression for the normal stress $\sigma_{\theta\theta}$ on the boundary surfaces Γ_ν is also necessary. Using relations (3.3) and (4.2), we can represent it in the form

$$\sigma_{\theta\theta} = |S_{\theta\theta}|\cos(\omega t - \Omega); \quad \Omega = -\arg S_{\theta\theta}, \quad S_{\theta\theta} = \sum_{m=1}^{\infty} S_{\theta\theta}^{(m)}\cos\lambda_m x_3$$

$$\begin{aligned} S_{\theta\theta}^{(m)} &= S_1^{(m)} - N^{(m)} = \mp\frac{i(y_{1m}(\zeta_0) + y_{2m}(\zeta_0))}{1-\nu} \\ &+ \int\left\{(y_{1m}(\zeta)e^{i(\psi-\alpha_0)} + y_{2m}(\zeta)e^{i(\alpha_0-\psi)})\left(\frac{\mu_{1m}}{2(1-\nu)}H_{11m}^0 - \lambda_m g_{1m}\right)\right. \\ &\left. + y_{3m}(\zeta_0)\lambda_m\left[\frac{1}{2(1-\nu)}H_{01m}^0 - H_{02m}^0 - \frac{\lambda_m^2}{\gamma_2^2}H^0\right]\right\}dS - N^{(m)}, \quad \zeta_0 \in \Gamma \end{aligned} \tag{4.5}$$

We will now consider the pulse excitation of a layer with a tunnel cavity. Introducing the Fourier integral transformation with respect to time

$$U_j(x, \omega)\frac{1}{\sqrt{2\pi}}\int_0^\infty u_j(x, t)e^{i\omega t}dt, \quad u_j|_{t=0} = \left.\frac{\partial u_j}{\partial t}\right|_{t=0} = 0$$

$$u_j(x, t) = \sqrt{\frac{2}{\pi}}\mathrm{Re}\int_0^\infty U_j(x, \omega)e^{-i\omega t}d\omega; \quad j = 1, 2, 3$$

we reduce boundary-value problem (1.7), (1.9) with respect to the Fourier transform of the corresponding densities to a system of integral Eqs. (4.4), where the right-hand sides now represent the spectral functions of the load acting on the cavity surface. The solution of the pulse problem is the superposition of "elementary" solutions over the whole frequency spectrum.

5. Some results

Suppose the layer is weakened by a tunnel cavity with a contour of the cross section in the form of an ellipse ($\zeta = R_1\cos\phi + iR_2\mathrm{sing}\phi$) or a square with rounded corners ($\zeta = R(e^{i\phi} + 0.14036e^{-3i\phi})$). A normal pressure, varying

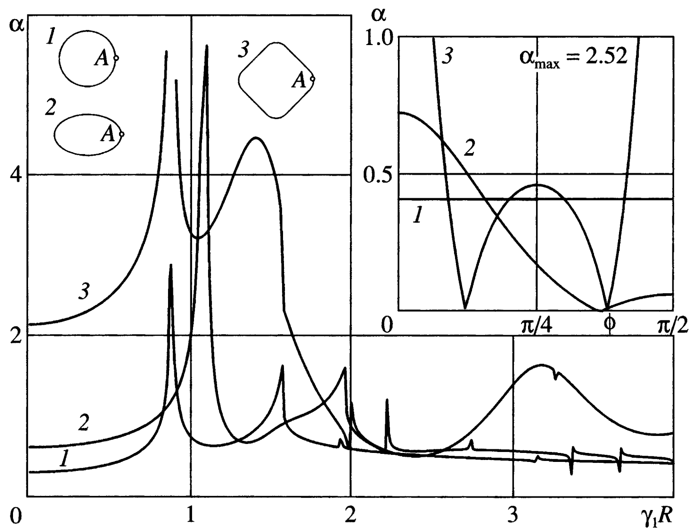

Fig. 1.

harmonically with time, having an amplitude

$$N = N_0 \cos\frac{\pi x_3}{2h} \; (N_0 = \text{const}) \tag{5.1}$$

acts on the cavity surface.

The calculations were carried out in the following sequence. We first found the approximate numerical solution of the system of integral Eqs. (4.4) using the method of mechanical quadratures,[13] and then, from relations (4.5), we established the amplitude values of the mechanical stresses $\sigma_{\theta\theta}$.

The change in the relative value of $\alpha = |S_{\theta\theta}/N_0|$ at the point A ($x_3 = 0$) as a function of the relative wave number $\gamma_1 R$ for a cavity of circular cross section ($R_1 = R_2 = R = 1$), elliptical cross section ($R_1 = 3/2$, $R_2 = 1$, $R = (R_1 + R_2)/2$) and "square" cross section ($R = 1$) is presented in Fig. 1. Here $h = 1$ and $\nu = 0.28$. Curves 1, 2 and 3 are constructed for a circle, ellipse and a square respectively. In the right upper corner of Fig. 1 we show the distribution of the quantity α along the circular, elliptic and square contours in the middle plane of the layer for the same parameters and the same correspondence as above, for $\gamma_1 R = 0.5$ (the distribution is symmetrical about the axis $\phi = \pi/2$).

We will now consider a layer weakened by two tunnel cavities. Curve 1 in Fig. 2 illustrates the distribution of the quantity α along the contour of the orifice (in the middle plane of the layer) for the case of two circular cavities of the same radius $R = 1$ with a bridge between them $d = 0.1$. Curve 2 corresponds to the distribution of the quantity α along the contour of an elliptical orifice ($R_1 = 3/2$ and $R_2 = 1$), interacting with a circular orifice ($R = 1$) when $d = 0.1$. In both cases a normal pressure of amplitude (5.1) acts on the surface of the cavity. The remaining parameters are the same as above.

In the right upper part of Fig. 2 we show the amplitude-frequency characteristic of the quantity α at the point A for a cylinder of finite length with a cross section in the form of a square with rounded corners for the same values of the parameters as above.

Suppose a trapeziform pressure pulse acts on the surface of a circular cavity (T is the pulse length)

$$N = N_0 \cos\frac{\pi x_3}{2h} \begin{cases} \tilde{t}, & 0 \le \tilde{t} \le 1 \\ 1, & 1 < \tilde{t} \le n - 1 \;\; (n \ge 2), \quad \tilde{t} = n\dfrac{t}{T} \\ (n - \tilde{t}), & n - 1 < \tilde{t} \le n \end{cases}$$

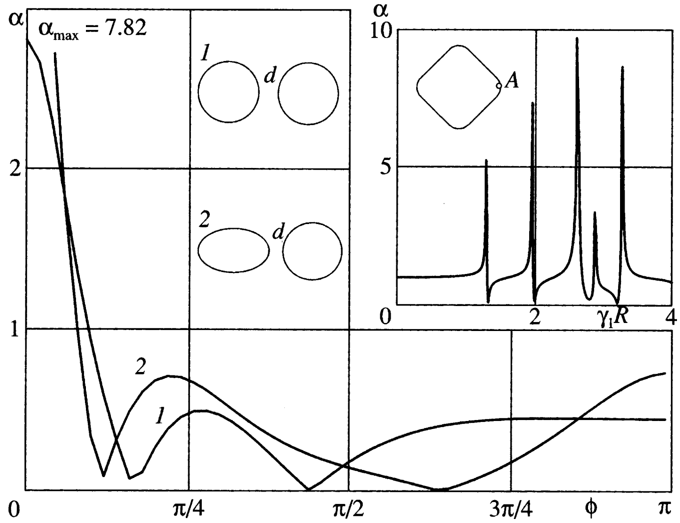

Fig. 2.

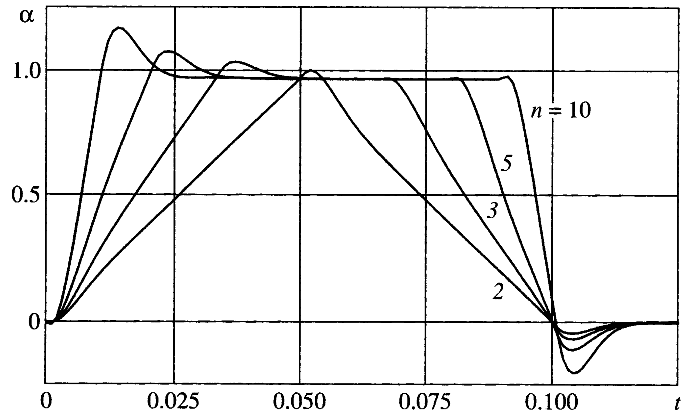

Fig. 3.

The results of calculations of the evolution of the relative quantity $\alpha = \sigma_{\theta\theta}/N_0$ with time for different values of n are shown in Fig. 3 for

$$T = 0.1, \quad h = 100, \quad x_3 = 0, \quad R = 10, \quad \nu = 0.28$$

Hence, we have developed a fairly effective method of solving three-dimensional boundary-value problems of the harmonic oscillations of a multiply connected cylindrical body with mixed boundary conditions on its bases.

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